ISRAEL JOURNAL OF MATHEMATICS, Vol. 33, Nos. 3–4, 1979

ON THE EXISTENCE OF σ-FINITE INVARIANT MEASURES FOR OPERATORS†

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To the Memory of Shlomo Horowitz

ABSTRACT

Several necessary and sufficient conditions are given for the existence of a σ-finite invariant measure for a positive operator on L_∞. They are of *σ-type*: the entire space is an increasing union of sets X_k each of which is well-behaved.

In a famous article [6], T. E. Harris gave a simple probabilistic condition for the existence of a σ-finite invariant measure for a discrete parameter Markov process with general state space. In fact, the Harris condition proved sufficient for the development of the theory of such processes, entirely analogous to the theory of discrete-state space processes (see in particular Orey's book [17]). On the other hand, an analytic translation of the Harris condition (cf. [9], [4], [10]) led to a parallel development of the operator ergodic theory of Harris processes, culminating in elegant books of Foguel [5] and Revuz [21]. Shlomo Horowitz contributed to this development the interesting papers [7], [8], and others.

The analytic formulation of the Harris condition consists in assuming that the Markov transition probability generates an irreducible conservative L_1 operator T, preserving the integral, hence necessarily a contraction, the adjoint of which $T^* = V$ possesses something of a beginning of a density: see the condition (NS) below. The assumption that T — or equivalently V — is a contraction was weakened by D. S. Ornstein and the second-named author [20] to

(B) $$\liminf | V^n h | < \infty \text{ a.e. for each } h \in L_\infty.$$

In the present paper we show that (B) is not necessary for the existence of an equivalent invariant measure, and we give several weaker conditions that are

† Research in part supported by the National Science Foundation (U.S.A.).
Received January 9, 1979

both necessary and sufficient. These conditions are of 'sigma' type, by which we mean that the entire space X is a countable increasing union of sets X_k each of which is in a certain sense well-behaved. Thus the condition below closest to (B) is (3') which asserts that for each k, $\liminf |V^n h| < \infty$ a.e. on X_k for each measurable function h 'beaten by 1_{X_k}', i.e., such that there exists integers K and N with $|h| \leqq K \Sigma_{i \leqq N} V^i 1_{X_k}$. Another necessary and sufficient condition is that for each k and each measurable set B contained in X_k, the ratio $1_B V^n 1_B / \Sigma_{i \leqq n} V^i 1_B$ converges to zero in measure; this can be approximately described as a condition sigma–C, where C is the Chacon–Ornstein lemma ([1], lemma 2). There is also below a condition sigma–C where C is a version of Orey's theorem ([16], [11]). It is easy to understand why such conditions are *necessary*: in presence of an infinite invariant measure, a change of measure renders the considered operator a contraction to which the Chacon–Ornstein lemma or Orey's theorem are applicable. In fact, we pass here through the dual operator, but the theory is entirely symmetric, and to every condition stated in terms of V there is a symmetrical one stated in terms of T. Also, the initial setting need not be the usual $L_1 - L_\infty$; we write the paper in terms of L_p-spaces $1 \leqq p \leqq \infty$, but the arguments are valid for larger classes of measure function spaces with an integral representation of linear functionals to which Fubini's theorem is applicable. The assumptions made on the operator: 'conservative' and 'irreducible' (= ergodic), are still sufficient, if properly defined, but they may be more difficult to verify. (These two notions together are called 'regular'.) Thus already in the case of L_2, in the absence of Hopf's maximal ergodic theorem, it is necessary to assume that T (or V) acts 'conservatively' on *each* positive function, while in L_1 this property is to be verified for only one function.

In [20] the sufficiency of (B) is shown using Ornstein's ergodic theorem [18], an abstract form of the Chacon–Ornstein theorem [1]. A modification of the argument in [20] could also prove the sufficiency of (3) and (3') below, but probably not that of other conditions. Our proofs are simple, but not 'constructive': we use Banach limits. It is almost certain that other, longer proofs could be found, which would be more 'constructive'.

Section 1 below gives definitions and establishes some technical lemmas. Section 2 proves the main theorem. Section 3 gives an example where the condition (B) fails but there exists an invariant measure.

1. Transition measures and small sets

Let X be an abstract set, $\mathscr{A}$ a countably generated σ-field of subsets of X, m a σ-finite measure on $\mathscr{A}$. All sets and functions appearing below are assumed measurable; they are considered equal if they are equal m-almost everywhere (a.e.); the words a.e. may or may not be omitted. Let $V(x, A)$ be a *transition measure*, i.e., a map $V: X \times \mathcal{A} \to R_+ \cup \{+\infty\}$ such that:

$$V(x, \cdot) \text{ is a positive } \sigma\text{-finite measure for each fixed } x \in X;$$

$$V(\cdot, A) \text{ is an } \mathcal{A}\text{-measurable function taking values}$$

$$\text{in } R_+ \cup \{+\infty\} \text{ for each fixed } A \in \mathcal{A}.$$

A *transition probability* is a transition measure such that $V(x, X) = 1$ for every x.

We assume that $V(x, A)$ is *null-preserving*, that is such that if $A \in \mathcal{A}$, then $m(A) = 0$ implies $V(x, A) = 0$ a.e. We also assume that there exists a number q, $1 \leqq q \leqq \infty$, such that for every $f \in L_q^+$, the function Vf defined by

$$Vf(x) = \int V(x, dy) f(y), \qquad x \in X,$$

is an element of L_q^+. Define inductively transition measures as follows:

$$V^0(x, A) = 1_A(x)$$

$$V^n(x, A) = \int V^{n-1}(x, dy) V(y, A), \qquad n = 1, 2, \cdots.$$

The transition measure V also induces an L_p operator T, where $1/p + 1/q = 1$: If $g \in L_p^+$, the measure γ on $\mathcal{A}$ defined by

$$\gamma(A) = \int g(x) V(x, A) m(dx)$$

is σ-finite; since $\gamma \ll m$, we may set $Tg = d\gamma/dm$. If $q = \infty$, $g \in L_1^+$, Tg is integrable because of the assumptions made on $V(x, A)$. If $1 \leqq q < \infty$, Tg defines in an obvious way a positive linear functional on L_q; hence $Tg \in L_p$. We say that the transition measure $V(x, A)$ induces the L_q operator V and the L_p operator T. Furthermore,

$$\forall f \in L_q, \quad \forall g \in L_p, \qquad \int (Vf) g\, dm = \int f\, Tg\, dm.$$

We still denote by V the extension of the operator V to the set of positive measurable functions $\mathcal{M}^+$, defined as follows: $V(\lim \nearrow f_n) = \lim \nearrow Vf_n$. Similarly T extends to $\mathcal{M}^+$. The duality relation still holds for the extended operators.

For each fixed $n > 0$ and x, write the Lebesgue decomposition of $V^n(x, \cdot)$

$$\forall A \in \mathcal{A}, \quad V^n(x, A) = \int_A d_n(x, y) m(dy) + V_s^n(x, A),$$

where $V_s^n(x, \cdot) \perp m$, and $d_n(x, y)$ is the n-step *density function*. We may and do assume that the densities d_n are jointly measurable and chosen so that for each x and y in X, and all integers n, m,

$$d_{n+m}(x, y) \geqq \int d_m(x, z) d_n(z, y) m(dz)$$

(see e.g., Orey [17]).

We assume that the transition measure $V(x, A)$ satisfies the following *non-singularity* assumption (NS):

(NS): There exists a non-null set G and an integer

$n_0 > 0$ such that for each $x \in G$, $m\{y \mid d_{n_0}(x, y) > 0\} > 0$.

If S is an operator, let $S_\infty = I + S + S^2 + \cdots$. An operator S on L_r is called *irreducible* if for any non-null f in L_r^+ one has $S_\infty f > 0$ a.e. Given $f \in L_r^+$, we denote by S_f the class of functions $h \in L_r$ which "can be beaten by f", i.e., such that $|h| \leqq \alpha \sum_{i<N} S^i f$, for some positive number α and some integer N. We write S_B for S_{1_B}. We call a function $e \in L_r^+$ *small for* S if $e \in S_f$ for every $f \in L_r^+$ with support intersecting the support of e. We call a set B *small for* S if 1_B is small for S.

We want to show that under the assumptions made on the transition measure $V(x, A)$, X is a union of sets small for the operators T and V. It is easy to see that T is irreducible if and only if V is irreducible; then also the transition measure $V(x, A)$ is called irreducible.

The following lemma gives a sufficient condition (in terms of densities) for two sets A and B to be small for the operators T and V.

1.1. Lemma. *Let $V(x, A)$ be an irreducible transition measure. Let A and B be non-null sets such that there exists a positive number a and an integer N satisfying*

$$\forall x \in A, \quad \forall y \in B, \qquad \sum_{i<N} d_i(x, y) \geqq a.$$

Then A is small for V and B is small for T.

Proof. Let $f \in L_q^+$ be a non-null function with support included in A, and let k be such that $f' = \inf(V^k f, 1_B) \not\equiv 0$. For any $x \in A$,

$$\begin{aligned}\sum_{i<N+k} V^i f(x) &\geqq \sum_{i<N} V^i f'(x)\\ &\geqq \int_B \Big[\sum_{i<N} d_i(x,y) f'(y)\Big] m(dy)\\ &\geqq a \int_B f'(y) m(dy) = \alpha > 0.\end{aligned}$$

Hence $\Sigma_{i<N+k} V^i f \geqq \alpha 1_A$, so that A is small for V. Let $g \in L_p^+$ be a non-null function with support included in B, and let M be such that $g' = \inf(T^M g, 1_A) \neq 0$. For any subset D of B,

$$\int_D \Big(\sum_{i<N+M} T^i g\Big) dm \geqq \int_D \Big(\sum_{i<N} T^i g'\Big) dm \geqq \int_X g'(x) \Big[\sum_{i<N} V^i(x,D)\Big] m(dx)$$

$$\geqq \int_A g'(x) \Big[\int_D \sum_{i<N} d_i(x,y) m(dy)\Big] m(dx) \geqq a m(D) \int_A g'(x) m(dx).$$

Since this inequality holds for every $D \subset B$, we deduce that $\Sigma_{i<N+M} T^i g \geqq a \int_A g'(x) m(dx)$ on B. This proves that B is small for T. □

Notice that the above proof shows that if B is a non-null set such that there exists a positive number a and an integer N satisfying $\Sigma_{i<N} d_i(x,y) \geqq a$ for every x and y in B, then B is small for T and V (without any irreducibility assumption).

1.2. Lemma. *Let $V(x,A)$ be an irreducible transition measure satisfying the* (NS) *condition. Then there exists a non-null set G such that for every $x \in G$, $\Sigma_i d_i(x,y) > 0$ a.e.*

Proof. Let G be the non-null set and n_0 the integer given in the (NS) condition. For all integers n and i and every x, the measure $A \to \int d_n(x,y) V^i(y,A) m(dy)$ is absolutely continuous with respect to m, and dominated by $V^{n+i}(x,\cdot)$; therefore

$$\int_A d_{n+i}(x,y) m(dy) \geqq \int d_n(x,y) V^i(y,A) m(dy), \qquad A \in \mathcal{A}.$$

We now have

$$\int_A \sum_{i<N} d_{n_0+i}(x,y) m(dy) \geqq \int d_{n_0}(x,y) \Big[\sum_{i<N} V^i(y,A)\Big] m(dy).$$

Let $m(A) > 0$ and $x \in G$; by (NS), $\int_A \Sigma_{i \geqq n_0} d_i(x,y) m(dy) > 0$. □

1.3. THEOREM. *Let $V(x,A)$ be an irreducible transition measure satisfying* (NS). *Then X can be represented as a countable disjoint union of sets X_i, such that each X_i is small for T and V.*

PROOF. Let G be the non-null set given by Lemma 1.2; we may and do assume $m(G)<\infty$. We will show that the assumptions of Lemma 1.1 are satisfied with $A=B$ a non-null subset of G. For any integer j and any $x\in G$, set

$$G_j(x)=\Big\{y\in G \,\Big|\, \sum_{i<j} d_i(x,y)>1/j\Big\}.$$

For any $x\in G$ there exists a smallest integer $i=i(x)$ such that $m[G_i(x)]\geqq (7/8)m(G)$. Set $E_j=\{x\in G \mid i(x)=j\}$. Since $G=\Sigma E_j$, there exists M such that $m(E_1+\cdots+E_M)\geqq(7/8)m(G)$; set $E=E_1+\cdots+E_M$. For $x\in E$, $G_{i(x)}(x)\subset G_M(x)$, hence $m[G_M(x)\cap E]\geqq m(E)-m(G\backslash G_M(x))\geqq(7/8)m(G)-(1/8)m(G)\geqq 3/4m\,(E)$. Let

$$H=\Big\{(x,y)\in E\times E \,\Big|\, \sum_{i<M} d_i(x,y)>1/M\Big\},$$

$$H_1(x)=\{y\in E \mid (x,y)\in H\},\qquad \text{for } x\in E,$$

$$H_2(y)=\{x\in E \mid (x,y)\in H\},\qquad \text{for } y\in E.$$

Since for every $x\in E$, $H_1(x)=G_M(x)\cap E$, $m(H_1(x))\geqq(3/4)m(E)$, and we have

$$(m\times m)(H)=\int_E m[H_1(x)]m(dx)\geqq\frac{3}{4}[m(E)]^2.$$

Let $B=\{y\in E \mid m[H_2(y)]\geqq(1/2)m(E)\}$; then

$$\begin{aligned}\frac{3}{4}[m(E)]^2&\leqq(m\times m)(H)\\&=\int_B m[H_2(y)]m(dy)+\int_{E\backslash B} m[H_2(y)]m(dy)\\&\leqq m(E)m(B)+\frac{1}{2}[m(E)-m(B)]m(E).\end{aligned}$$

Hence $m(B)\geqq(1/2)m(E)>0$. Furthermore, if $x\in B$ and $y\in B$, then $m[H_1(x)]\geqq(3/4)m(E)$, and $m[H_2(y)]\geqq(1/2)m(E)$, so that $m[H_1(x)\cap H_2(y)]\geqq(1/4)m(E)$. Hence if $x\in B$ and $y\in B$,

$$\sum_{i<2M} d_i(x,y) \geqq \frac{1}{2M-1}\int_{H_1^{1}(x)\cap H_2(y)} \Big\{\sum_{i<M} d_i(x,z)\Big\}\Big\{\sum_{i<M} d_i(z,y)\Big\} m(dz)$$

$$\geqq \frac{1}{2M-1}\frac{1}{M^2}\frac{m(E)}{4} > 0.$$

We deduce from Lemma 1.1 that B is a small set for T and V.

Since B is small for T, given any n, $\Sigma_{i<n}T^i 1_B$ is a small function for T; hence $B_n = \{\Sigma_{i<n}T^i 1_B \geqq 1/n\}$ is a small set for T. Since T is irreducible, $X = \lim\nearrow B_n$. Similarly, $X = \lim\nearrow C_n$, where each set C_n is a set small for V. Hence $X = \lim\nearrow(B_n \cap C_n)$, where each set $B_n \cap C_n$ is small for both T and V. □

For any set E and any r, set $L_r(E) = \{f \mid \operatorname{supp} f \subset E\} \cap L_r$. We say that a set E is *absorbing under* an L_r operator S if $f \in L_r(E)$ implies $Sf \in L_r(E)$.

1.4. Lemma. *Let T and V be the L_p and L_q operators induced by the transition measure $V(x,A)$. For every $f \in L_p^+$, $\{T_\infty f = \infty\}$ is absorbing under T, and for every $g \in L_q^+, \{V_\infty g = \infty\}$ is absorbing under V.*

Proof. We prove that if $f \in L_p^+$, $\{T_\infty f < \infty\}$ is absorbing under V. Indeed, let $g \in L_q^+(A_k) \cap L_1^+(A_k)$, where $A_k = \{T_\infty f \leqq k\}$. Then

$$\int VgT_\infty f dm = \int g \sum_{n\geqq 1} T^n f dm \leqq k \int g dm,$$

which clearly implies that supp $Vg \subset \{T_\infty f < \infty\}$. Since every function of $L_q^+(A_k)$ is an increasing limit of functions of $L_q^+(A_k) \cap L_1^+(A_k)$, and since $A_k = \{T_\infty f < \infty\}$, it follows that $\{T_\infty f < \infty\}$ is absorbing under V. Let $h \in L_p(\{T_\infty f = \infty\})$; since for any $g \in L_q(\{T_\infty f < \infty\})$, $\int (Th) g\, dm = \int h(Vg) dm = 0$, it follows that $\operatorname{supp}(Th) \subset \{T_\infty f = \infty\}$. Similarly one shows that $\{V_\infty g = \infty\}$ is absorbing under V. □

An L_r-operator S is called *conservative* if for any non-null function $f \in L_r^+$, $\{S_\infty f = 0 \text{ or } \infty\} = X$; S is called *dissipative* if there exists a non-null function $f \in L_r^+$ such that $\{S_\infty f < \infty\} = X$. Let $V(x,A)$ be a transition measure inducing irreducible operators T on L_p and V on L_q; then either V and T are both dissipative, or V and T are both conservative. Indeed, assume that V is irreducible and dissipative; let $g \in L_q^+$ be such that $X = \{0 < V_\infty g < \infty\}$. Let $h = V_\infty g$; clearly $Vh \leqq h$. Choose $f \in L_p^+$ such that $0 < \int fhdm < \infty$; then $\int f V_\infty g\, dm = \int (T_\infty f) g\, dm < \infty$ implies that $\{T_\infty f < \infty\} \supset \text{support}\, g$. Since $\{T_\infty f < \infty\}$ is absorbing under V, we have $\{T_\infty f < \infty\} = X$, which proves that T is dissipative. A similar argument shows that if T is irreducible and dissipative, so is V. Furthermore, an irreducible operator is either conservative or dissipative. If

V and T are both irreducible and conservative, then $V(x, A)$, V and T are called *regular.*

Let $\mathscr{R}$ be a ring of sets; a finite, non-negative, finitely additive set-function on $\mathscr{R}$ is called a *charge.* A charge λ on $\mathscr{R}$ is called a *pure charge* if λ does not dominate any non-trivial measure μ on $\mathscr{R}$.

A charge λ on a ring admits a unique decomposition (Yosida–Hewitt decomposition [23]) $\lambda = \mu + \pi$, where μ is a measure and π is a pure charge; to see this, set

$$\mu(A) = \inf\left\{\sum_{i=1}^{\infty} \lambda(A_i) \,\middle|\, A = \sum_{i=1}^{\infty} A_i,\ A_i \in \mathscr{R}\right\}.$$

It is known, and will be used below, that a pure charge on a σ-algebra $\mathscr{A}$ is ε-singular with respect to every measure μ: $\forall \varepsilon > 0$, $\exists C \in \mathscr{A}$ such that $\pi(X) = \pi(C)$ and $m(C) < \varepsilon$ (cf. [23] and e.g. [22]).

The following lemma shows how the existence of a subinvariant (invariant) positive function for the operator V can be deduced from the existence of a charge subinvariant (invariant) under T. A similar lemma is obtained exchanging the roles of V and T.

1.5. LEMMA. *Let $V(x, A)$ be an irreducible transition measure. Let B be a fixed non-null set of finite measure, and denote by $\mathscr{R}$ the ring of sets A such that $1_A \in T_B$. Suppose that Λ is a positive linear functional defined on T_B, such that:*

(a) $\forall A \in \mathscr{R}$, $\Lambda(T1_A) \leqq \Lambda(1_A)$;

(b) $\forall A \in \mathscr{R}, m(A) = 0 \Rightarrow \Lambda(1_A) = 0$;

(c) $\forall A \subset B$, $m(A) = 0 \Leftrightarrow \Lambda(1_A) = 0$.

Then there exists a function g such that $0 < g < \infty$ a.e., and $Vg \leqq g$. If $V(x, A)$ is regular, then $Vg = g$.

PROOF. Let λ be the charge defined on $\mathscr{R}$ by $\lambda(A) = \Lambda(1_A)$, and let $\lambda = \mu + \pi$ be the decomposition of λ into a measure and a pure charge. Notice that since T is irreducible, for every set $A \in \mathscr{A}$, $A = \lim\nearrow A \cap B_n$, where the sets $B_n = \{1/n \leqq \Sigma_{i<n} T^i 1_B\}$ are in $\mathscr{R}$. Hence $\mathscr{R}$ generates $\mathscr{A}$. Still denote by μ the unique extension of μ to the σ-algebra $\mathscr{A}$, and let $\tilde{\mu}$ be the measure defined on $\mathscr{A}$ by $\tilde{\mu}(A) = \int T1_A d\mu$. Given any $A \in \mathscr{R}$, there exists a sequence $f_n \nearrow T1_A$ of positive $\mathscr{R}$-measurable step functions. Then

$$\tilde{\mu}(A) = \lim\nearrow \int f_n d\mu \leqq \lim\nearrow \int f_n d\lambda \leqq \Lambda(T1_A) \leqq \lambda(A) = \mu(A) + \pi(A).$$

Hence the pure charge π dominates the positive measure $(\tilde{\mu} - \mu)^+$; this implies

$\tilde{\mu} \leqq \mu$ on $\mathcal{R}$, and hence $\tilde{\mu} \leqq \mu$ on $\mathcal{A}$. Condition (b) implies $\mu \ll m$. Since $X = \lim \nearrow B_n$, μ is σ-finite; set $g = d\mu/dm$. For every $A \in \mathcal{A}$,

$$\int_A Vgdm = \int gT1_A dm = \int T1_A d\mu \leqq \mu(A) = \int_A gdm;$$

hence $Vg \leqq g$. It remains to show that $g > 0$ a.e. We at first prove that the restrictions of μ and m to B are equivalent; let $\mathcal{B}$ be the σ-algebra of subsets of B. Assume that m is *not* absolutely continuous with respect to μ on $\mathcal{B}$, and let $A \in \mathcal{B}$ be such that $\mu(A) = 0$ and $m(A) > 0$. Given ε, $0 < \varepsilon < m(A)$, choose $C \in \mathcal{B}$ such that $m(C) < \varepsilon$ and $\pi(B) = \pi(C)$. Then $m(A \setminus C) > 0$ and $\lambda(A \setminus C) = \mu(A \setminus C) + \pi(A \setminus C) = 0$; this contradicts property (c), hence $B \subset \{g > 0\}$. Since $\forall f \in L_p^+$, $\int Tfgdm = \int fVgdm \leqq \int fgdm$, the set $\{g = 0\}$ is absorbing under the irreducible operator T; hence $\{g = 0\} \neq X$ implies $g > 0$ a.e.

Assume that V is conservative; then if $h = g - Vg$ is a non-null function, and $h' \in L_q^+$, $0 \leqq h' \leqq h$, one has for every n

$$\sum_{i<n} V^i h' \leqq \sum_{i<n} V^i h = g - V^n g \leqq g < \infty \quad \text{a.e.}$$

It follows that $g = Vg$. □

2. Existence of invariant measures

In this section we assume that the operator V (hence T) is regular, i.e., irreducible and conservative. We give necessary and sufficient conditions for the existence of an "equivalent invariant measure". We write $\overline{T^n}$ for $(1/n)\Sigma_{i<n}T^i$. Recall that V_A is the space of functions that "can be beaten by 1_A" — precise definition is given above.

2.1. THEOREM. *Let $V(x, A)$ be a null-preserving regular transition measure satisfying the* (NS) *condition. The following conditions are equivalent*:

(1) *There exists a sequence of sets $X_k \nearrow X$ such that for each k the sequence $(1_{X_k} T^n 1_{X_k})_n$ is uniformly integrable.*

(1′) *There exists a sequence of sets $X_k \nearrow X$ such that for each k the sequence $(1_{X_k} V^n 1_{X_k})_n$ is uniformly integrable.*

(2) *There exists a sequence of sets $X_k \nearrow X$ such that for each k the sequence $(1_{X_k} \overline{T^n} 1_{X_k})_n$ is uniformly integrable.*

(2′) *There exists a sequence of sets $X_k \nearrow X$ such that for each k the sequence $(1_{X_k} \overline{V^n} 1_{X_k})_n$ is uniformly integrable.*

(3) *There exists a sequence of sets* $X_k \nearrow X$ *such that for each* k *and each* $f \in T_{X_k}$, $\liminf |T^n f| < \infty$ *a.e. on* X_k.

(3′) *There exists a sequence of sets* $X_k \nearrow X$ *such that for each* k *and each* $f \in V_{X_k}$, $\liminf |V^n f| < \infty$ *a.e. on* X_k.

(4) *There exists a sequence of sets* $X_k \nearrow X$ *such that for each* k *and each* $B \subset X_k$,

$$\frac{1_B T^n 1_B}{\sum_{i \leq n} T^i 1_B}$$

converges to zero in measure.

(4′) *There exists a sequence of sets* $X_k \nearrow X$ *such that for each* k *and each* $B \subset X_k$,

$$\frac{1_B V^n 1_B}{\sum_{i \leq n} V^i 1_B}$$

converges to zero in measure.

(5) *There exists a sequence of sets* $X_k \nearrow X$ *and an integer* $\delta > 0$ *such that for each* k *and each* $B \subset X_k$,

$$\lim \| 1_{X_k} (T^n 1_B - T^{n+\delta} 1_B) \|_1 = 0.$$

(5′) *There exists a sequence of sets* $X_k \nearrow X$ *and an integer* $\delta > 0$ *such that for each* k *and each* $B \subset X_k$,

$$\lim \| 1_{X_k} (V^n 1_B - V^{n+\delta} 1_B) \|_1 = 0.$$

(6) *There exists a positive measurable function* u, $0 < u < \infty$, *such that* $Tu = u$.

(6′) *There exists a positive measurable function* u, $0 < u < \infty$, *such that* $Vu = u$.

Proof. The chain of implications is: (1) ⇒ (2) ⇒ (3) ⇒ (6′) ⇒ (1′) ⇒ (2′) ⇒ (3′) ⇒ (6) ⇒ (1), (4) ⇔ (6′) ⇔ (5), and (4′) ⇔ (6) ⇔ (5′). Obviously (1) ⇒ (2) and (1′) ⇒ (2′). Because of the symmetry in the statements and proofs, we will only show (2) ⇒ (3), (6) ⇒ (1), (6) ⇒ (4′) and (6′) ⇒ (5). Then to give a unified proof of implications (3) ⇒ (6′), (4) ⇒ (6′) and (5) ⇒ (6′), we will prove that under one of the conditions (3), (4) and (5), there exists a non-null small set B such that $1_B T^n 1_B / \Sigma_{i \leq n} T^i 1_B \to 0$ in measure, which in turn implies (6′) as shown in Lemma 2.3.

(2) $\Rightarrow$ (3):

2.2. LEMMA. *Let (a_n) be a sequence of positive numbers; then for every integer N,*

$$\liminf_n \frac{1}{N}\sum_{j=1}^{N} a_{n+j} \leqq \liminf_n \frac{1}{n}\sum_{i=1}^{n} a_i.$$

PROOF OF LEMMA. Indeed, let $a < b < \lim_n \inf (1/N)\Sigma_{j=1}^{N} a_{n+j}$. There exists n_0 such that $(1/N)\Sigma_{j=1}^{N} a_{n+j} \geqq b$ if $n \geqq n_0 N$. For every n, write $n = Nq_n + r_n$ where q_n, r_n are integers and $0 \leqq r_n < N$. Then

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geqq \frac{1}{N(q_n+1)}\sum_{j=n_0}^{q_n-1}(a_{Nj+1}+\cdots+a_{Nj+N}) \geqq \frac{q_n-n_0}{q_n+1}b \geqq a$$

for large values of n.

Hence $\lim_n \inf (1/n)\Sigma_{i=1}^{n} a_i \geqq a$, which implies the desired inequality. $\square$

Let $f \in T_{X_k}$, $|f| \leqq K\,\Sigma_{i\leqq N} T^i 1_{X_k}$; then by Lemma 2.2,

$$\begin{aligned}\liminf |T^n f| &\leqq K(N+1)\liminf T^n\Big(\frac{1}{N+1}\sum_{i\leqq N} T^i 1_{X_k}\Big)\\ &\leqq K(N+1)\liminf \overline{T^n} 1_{X_k} < \infty \quad \text{a.e. on } X_k,\end{aligned}$$

the last inequality following from (2) by Fatou's lemma.

(6) $\Rightarrow$ (1): Since m is σ-finite, there exists a sequence of sets A_k, such that $A_k \nearrow X$, and $m(A_k) < \infty$ for all k. Let u be as in (6), and set $X_k = A_k \cap \{1/k \leqq u \leqq k\}$. Then $X_k \nearrow X$; furthermore

$$1_{X_k} T^n 1_{X_k} \leqq k\, 1_{X_k} T^n u \leqq k u\, 1_{X_k} \leqq k^2 1_{A_k}.$$

(6) $\Rightarrow$ (4'): Let $0 < u = Tu \in \mathcal{M}^+$. Define a transition probability P by

$$\forall x \in X, \quad \forall A \in \mathcal{A}, \quad P(x,A) = \frac{1}{u(x)} T(u 1_A)(x).$$

For every $f \in L_1$ and $g \in L_\infty$, denote by fP the L_1-action and by Pg the L_∞-action induced by P. For every function $f \in L_1$ and every $A \in \mathcal{A}$,

$$\begin{aligned}\int_A fP\,dm &= \int f(x)P(x,A)\,m(dx)\\ &= \int \frac{f(x)}{u(x)} T(u 1_A)(x)\,m(dx)\\ &= \int_A u(x)\, V\Big(\frac{f}{u}\Big)(x)\,m(dx).\end{aligned}$$

Hence $fP = u\,V(f/u)$, so that for every k, $fP^k = uV^k(f/u)$. Let $Y_k \nearrow X$ be a sequence of sets of finite measure. Set $X_k = Y_k \cap \{u \leqq k\}$, and let $B \subset X_k$ for some fixed k. Then $f = u\,1_B \in L_1$; by the Chacon–Ornstein lemma ([1], lemma 2; or [5] p. 22), $fP^n/\Sigma_{i\leqq n} fP^i = V^n 1_B/\Sigma_{i\leqq n} V^i 1_B$ converges to zero a.e. on B.

(6′) $\Rightarrow$ (5): Let $0 < u = Vu \in \mathcal{M}^+$ and $P(x, A) = u(x)^{-1}V(u\,1_A)(x)$. Then $P(x, A)$ inherits the properties of the transition measure $V(x, A)$: $P(x, A)$ is regular and satisfies the condition (NS). Indeed, let f be a non-null element of L_1^+; choose $g \leqq f$, $g \neq 0$ such that $g/u \in L_p^+$. Then $T_\infty(f/u) \geqq T_\infty(g/u) = \infty$ a.e. implies that $fP_\infty = \infty$ a.e. Since the transition measure $V(x, A)$ satisfies (NS), there exists a non-null set G and an integer n_0 such that $m\{y \mid d_{n_0}(x, y) > 0\} > 0$. Since for any integer n,

$$P^n(x, A) = \int_A \frac{d_n(x, y)}{u(x)}\, u(y) m(dy) + \frac{1}{u(x)} \int_A u(y) V_s^n(x, dy),$$

with $V_s^n(x, \cdot) \perp m$, the n-step density of P is $d_n(x, y)u(y)/u(x)$; thus P also satisfies (NS). Hence P has a *period* δ (see [17], [5], or [13]). Let $X = C_1 + \cdots + C_\delta$, every C_i is absorbing under the L_1 operator P^δ, the restriction of P^δ to C_i is irreducible, and the L_1-operator P carries C_1 on $C_2, \cdots, C_{\delta-1}$ and C_δ and C_δ on C_1. The set G intersects at least one of the sets C_i, say C_1; set

$$K(x, A) = P^\delta(x, A), \qquad \forall x \in C_1, \quad \forall A \subset C_1.$$

Then $K(x, A)$ is a transition probability on C_1; if $f \in L_\infty(C_1)$, denote $Kf = \int K(x, dy) f(y)$, and if $g \in L_1(C_1)$, and $\gamma(A) = \int g(x) K(x, A) m(dx)$, $A \subset C_1$, set $gK = d\gamma/dm$. Every power of $K(x, A)$ is irreducible.

We now check that $K(x, A)$ satisfies the *essential Harris condition* (EH), i.e., for each null-set $N \subset C_1$, there exists a point $x \in C_1 \backslash N$ and an integer $n > 0$ such that the measure $K^n(x, \cdot)$ is not singular with respect to m (see e.g. [5]).

Notice that (NS) and (EH) are equivalent. Indeed if (NS) holds, then for any null-set N, every $x \in G \cap N^c$ is such that $K^{n_0}(x, \cdot)$ is not singular with respect to m. Conversely let $N = \{x \mid \exists n, K^n(x, \cdot)$ is not singular w.r. to $m\}$. Since the densities are jointly measurable, it is easy to see that N is measurable. If N were a null-set, then by (EH) there would exist $x \notin N$ and an integer n such that $K^n(x, \cdot)$ is not singular with respect to m. This contradicts the definition of N. Let $p_n(x, y)$ be the n step density function of $P(x, A)$. By (NS), given any null subset N of C_1, since $C_1 \cap G$ is non-null, there exists $x \in (C_1 \cap G)\backslash N$ such that $\{y \mid p_{n_0}(x, y) > 0\}$ is non-null. Hence there exists $a > 0$ and a non-null set A of finite measure included in $\{y \mid p_{n_0}(x, y) \geqq a\}$. Since P is irreducible, there exists n

such that $\inf(1_A P^n, 1_{C_1}) \neq 0$. Let B be a non-null subset of C_1 such that for some $b > 0$, $b 1_B \leqq 1_A P^n$. Then

$$\int 1_B(y) P^{n_0+n}(x, dy) \geqq \int p_{n_0}(x,y) P^n 1_B(y) m(dy)$$

$$\geqq a \int_A (P^n 1_B)(y) m(dy) = a \int_B (1_A P^n)(y) m(dy) \geqq abm(B) > 0.$$

Since $x \in C_1$ and $B \subset C_1$, $n_0 + n$ is a multiple of the period δ; hence K acting on C_1 satisfies the condition (EH). Let $f \in L_1(C_1)$ be such that $\int f dm = 0$; then by Orey's theorem, $\| fK^n \|_1 \to 0$ (see [17], and [19] theorem 4.1). For any set $B \subset C_1 \cap \{u \leqq k\}$ of finite measure, set $f = u(T^\delta 1_B - 1_B)$ where $0 < u = Vu$. Then

$$\int |f| dm \leqq \int u(T^\delta 1_B) dm + \int u 1_B dm = 2 \int u 1_B dm \leqq 2km(B).$$

Furthermore, $\int f dm = \int u(T^\delta 1_B) dm - \int u 1_B dm = 0$. Hence

$$\| uT^{n\delta}(T^\delta 1_B - 1_B)\|_1 = \| fP^{n\delta}\|_1 \to 0.$$

Given any $i = 1, \cdots, \delta$ and any set B of finite measure, $B \subset C_i \cap \{u \leqq k\}$, set $f = u(T^{2\delta - i+1} 1_B - T^{\delta - i + 1} 1_B)$; $f \in L_1(C_1)$ and $\int f dm = 0$. Hence for any $j = 0, \cdots, \delta - 1$,

$$\| u(T^{(n+2)\delta + j} 1_B - T^{(n+1)\delta + j} 1_B)\|_1 = \| fP^{n\delta + i - 1 + j}\|_1 \leqq \| fP^{n\delta}\|_1 \to 0.$$

Since $\sum_{i=1}^{\delta} C_i = X$, one has that for every set of finite measure $B \subset \{u \leqq k\}$ and every $j = 0, \cdots, \delta - 1$, $\| u(T^{(n+i)\delta + j} 1_B - T^{n\delta + j} 1_B)\|_1 \to 0$. Let $Y_k \nearrow X$ be a sequence of sets of finite measure, and set $X_k = \{1/k \leqq u \leqq k\} \cap Y_k$. Then $X_k \nearrow X$ and for every $j = 0, \cdots, \delta - 1$, from $\| u(T^{(n+1)\delta + j} 1_{X_k} - T^{n\delta + j} 1_{X_k})\|_1 \to 0$, we deduce $\| 1_{X_k}(T^{(n+1)\delta + j} 1_{X_k} - T^{n\delta + j} 1_{X_k})\|_1 \to 0$.

2.3. Lemma. *Let $V(x, A)$ be an irreducible transition measure. Let B be a non-null set small for T and such that*

$$\frac{1_B T^n 1_B}{\sum_{i \leqq n} T^i 1_B} \quad \textit{converges to } 0 \textit{ in measure.} \tag{2.4}$$

Then there exists a function u such that $0 < u < \infty$ and $Vu \leqq u$. If $V(x, A)$ is regular, then $Vu = u$.

Proof of Lemma 2.3. Let L be a Banach limit (see e.g. [3]). Let $f \in T_B$, $f \geqq 0$;

choose K and N fixed and such that $f \leqq K\,\Sigma_{i\leqq N} T^i 1_B$. For every $j \geqq N$, developing the expression $\Sigma_{i\leqq n-j} T^i(\Sigma_{r\leqq N} T^r 1_B)$ one obtains at most $N+1$ terms $T^k 1_B$ for a fixed k; therefore

$$\sum_{i\leqq n-j} T^i f \leqq K(N+1) \sum_{k\leqq n-j+N} T^k 1_B \leqq K(N+1) \sum_{i\leqq n} T^i 1_B. \tag{2.5}$$

It follows that $(\int_B [\Sigma_{i\leqq n-j} T^i f / \Sigma_{i\leqq n} T^i 1_B] dm)$ is bounded. Set

$$\Lambda_j(f) = L\left(\int_B \frac{\sum_{i\leqq n-j} T^i f}{\sum_{i\leqq n} T^i 1_B} dm\right), \qquad j \geqq N;$$

we show that $\Lambda_j(f) = \Lambda_N(f)$. Indeed, clearly $\Lambda_j(f) \leqq \Lambda_N(f)$. Conversely, the ratio in (2.4), being bounded by 1_B, converges to 0 also in L_1. Hence

$$\Lambda_N(f) = L\left(\int_B \frac{\sum_{i\leqq n-j} T^i f}{\sum_{i\leqq n} T^i 1_B} dm + \int_B \frac{\sum_{n-j+1\leqq i\leqq n-N} T^i f}{\sum_{i\leqq n} T^i 1_B} dm\right),$$

and by a computation similar to (2.5),

$$\int_B \frac{\sum_{n-j+1\leqq i\leqq n-N} T^i f}{\sum_{i\leqq n} T^i 1_B} dm \leqq K(N+1) \sum_{0\leqq i\leqq j-1} \int_B \frac{T^{n-i} 1_B}{\sum_{k\leqq n} T^k 1_B},$$

which converges to zero. Set for positive $f \in T_B$, $\Lambda f = \Lambda_N(f)$, and extend Λ by linearity. Λ is a positive linear functional on T_B; we will show that Λ satisfies the assumptions of Lemma 1.5. Let A be a set such that $1_A \in T_B$, $1_A \leqq K\,\Sigma_{i\leqq N} T^i 1_B$. Then

$$\begin{aligned}
\Lambda(T1_A) &= \Lambda_{N+1}(T1_A) \\
&= L\left(\int_B \frac{\sum_{i\leqq n-N-1} T^i(T1_A)}{\sum_{i\leqq n} T^i 1_B} dm\right) \\
&= L\left(\int_B \frac{\sum_{i\leqq n-N} T^i 1_A}{\sum_{i\leqq n} T^i 1_B} dm - \int_B \frac{1_A}{\sum_{i\leqq n} T^i 1_B} dm\right) \\
&\leqq \Lambda_N(1_A) \\
&= \Lambda(1_A).
\end{aligned}$$

Obviously $m(A)=0$ implies $\Lambda(1_A)=0$. Let $A\subset B$ be a non-null set; since B is small for T, there exist K and N such that $1_B \leqq K\,\Sigma_{i\leqq N}T^i 1_A$. Since $0<m(B)=\Lambda(1_B)\leqq K\Sigma_{i\leqq N}\Lambda(T^i1_A)\leqq K(N+1)\Lambda(1_A)$, the assumptions of Lemma 1.5 are satisfied. □

(3) ⇒ (6′): By Theorem 1.3 there exists a small set A. Assume (3); one of the sets $X_i \cap A = B$ is non-null. The set B is small and such that for every $f\in T_B$, $\liminf|T^nf|<\infty$ on B. We will show that

$$\lim 1_B \frac{T^n 1_B}{\sum_{i\leqq n} T^i 1_B}=0 \qquad \text{a.e.}$$

Otherwise there would exist $\varepsilon>0$ and a non-null set $A\subset B$ such that $T^n1_B>\varepsilon\,\Sigma_{i\leqq n}T^i1_B$ infinitely often on A. The proof is now in part similar to that of lemma 2 [1]. For every $n\geqq 0$, set

$$a_n = T^n 1_B - \varepsilon\sum_{i\leqq n} T^i 1_B, \qquad A_n=\{a_n>0\}\cap B.$$

Then $a_{n+1}+\varepsilon 1_B = Ta_n \leqq Ta_n^+$ implies $a_{n+1}^+ + \varepsilon\, 1_{A_{n+1}} = 1_{A_{n+1}}(a_{n+1}+\varepsilon)\leqq Ta_n^+$. Summing,

$$\varepsilon\sum_{n\leqq N} 1_{A_{n+1}} \leqq -\sum_{n\leqq N} a_n^+ + \sum_{n\leqq N} Ta_n^+ + (1-\varepsilon)1_B.$$

Since $A\subset\limsup A_n$, $\Sigma_{n\leqq N}1_{A_{n+1}}\to\infty$ on A. By Egorov's theorem the convergence is uniform on a non-null subset $A'\subset A$. Since B is a small set, there exist α and K such that $1_B\leqq\alpha\,\Sigma_{j\leqq K}T^j1_{A'}$. Given M, choose N so big that $M1_{A'}\leqq\varepsilon\Sigma_{n\leqq N}1_{A_{n+1}}$, and set $f=\Sigma_{i\leqq N}a_i^+$. Then

$$M1_{A'}\leqq\varepsilon\sum_{n\leqq N}1_{A_{n+1}}\leqq Tf-f+(1-\varepsilon)1_B\leqq Tf-f+(1-\varepsilon)\alpha\sum_{j\leqq K}T^j1_{A'}.$$

Applying $\Sigma_{i\leqq n}T^i$, we obtain (cf. (2.5))

$$\begin{aligned} M\sum_{i\leqq n}T^i1_{A'} &\leqq T^{n+1}f-f+\alpha(1-\varepsilon)\sum_{i\leqq n}T^i\Big(\sum_{j\leqq K}T^j1_{A'}\Big)\\ &\leqq T^{n+1}f+\alpha(1-\varepsilon)(K+1)\sum_{i\leqq n+K}T^i1_{A'}.\end{aligned}$$

Choose $M>\alpha(1-\varepsilon)(K+1)$; then

$$\begin{aligned} 0 &\leqq [M-\alpha(1-\varepsilon)(K+1)]\sum_{i\leqq n}T^i1_{A'}\\ &\leqq T^{n+1}f+\alpha(1-\varepsilon)(K+1)\sum_{i=n+1}^{n+K}T^i1_{A'}\\ &= T^{n+1}\Big[f+\alpha(1-\varepsilon)(K+1)\sum_{i\leqq K-1}T^i1_{A'}\Big].\end{aligned}$$

This brings a contradiction since $\Sigma_{i\leqq n} T^i 1_{A'} \to \infty$ a.e., and

$$f + \alpha(1-\varepsilon)(K+1) \sum_{i \leqq K-1} T^i 1_{A'} \in T_B.$$

Hence the assumptions of contradiction 2.3 are satisfied, which implies the existence of $0 < u < \infty$ such that $Vu = u$.

(4) $\Rightarrow$ (6'): As in the proof of the previous implication, we may and do assume that B is a non-null set small for T such that $B \subset X_k$ for some k. The assumptions of Lemma 2.3 are clearly satisfied by B, which implies the existence of u, such that $Vu = u$ and $0 < u < \infty$.

(5) $\Rightarrow$ (6'): As in the previous implications, we may and do assume that B is a non-null set small for T, and $B \subset X_k$ for some k. We will show that the condition (2.4) is satisfied by B. Assume not; there exists $\alpha > 0$ and an infinite set D of positive integers such that

$$m\left[B \cap \left\{T^n 1_B \geqq \alpha \sum_{i \leqq n} T^i 1_B\right\}\right] \geqq \alpha \qquad \text{for } n \in D. \tag{2.6}$$

Fix K and $\beta > 0$, and choose $N(K, \beta) = N$ such that

$$\forall n \geqq N, \quad \forall k \leqq K, \qquad \| 1_B (T^{n-k\delta} 1_B - T^n 1_B) \|_1 \leqq \beta.$$

Then, using Chebyshev's inequality,

$$m\left[B \cap \left\{| T^{n-k\delta} 1_B - T^n 1_B | \geqq \frac{\alpha}{2} \sum_{i \leqq n} T^i 1_B\right\}\right] \leqq m\left[B \cap \left\{| T^{n-k\delta} 1_B - T^n 1_B | \geqq \frac{\alpha}{2}\right\}\right]$$

$$\leqq \frac{2\beta}{\alpha}.$$

Let $n \geqq N$, $n \in D$; then

$$m\left[\bigcup_{k=1}^{K} \left(B \cap \left\{T^n 1_B \geqq \alpha \sum_{i \leqq n} T^i 1_B\right\} \cap \left\{T^{n-k\delta} 1_B < \frac{\alpha}{2} \sum_{i \leqq n} T^i 1_B\right\}\right)\right] \leqq \frac{2K\beta}{\alpha}.$$

Hence, using also (2.6), we have

$$\begin{aligned} & m\left[B \cap \bigcap_{k=1}^{K} \left\{T^{n-k\delta} 1_B \geqq \frac{\alpha}{2} \sum_{i \leqq n} T^i 1_B\right\}\right] \\ (2.7) \qquad & \geqq m\left[\bigcap_{k=1}^{K} \left\{T^{n-k\delta} 1_B \geqq \frac{\alpha}{2} \sum_{i \leqq n} T^i 1_B\right\} \cap B \cap \left\{T^n 1_B \geqq \alpha \sum_{i \leqq n} T^i 1_B\right\}\right] \\ & \geqq \alpha - \frac{2K\beta}{\alpha}. \end{aligned}$$

Given $\varepsilon > 0$, choose K such that $K\alpha/2 > 1$; then choose β such that $(2K\beta/\alpha) < \alpha/2$, and let $n \in D$ be larger than $N(K,\beta)$. Then by (2.7) on a subset of B of measure larger than $\alpha - (2K\beta/\alpha) \geqq \alpha/2 > 0$, one has

$$\sum_{k=1}^{K} T^{n-k\delta} 1_B \geqq K\frac{\alpha}{2} \sum_{i \leqq n} T^i 1_B > \sum_{i \leqq n} T^i 1_B,$$

which is a contradiction. □

REMARK. In Theorem 2.1 conditions 1–5′ can be replaced by equivalent conditions expressed in terms of non-null small sets. For instance, condition (4) is replaced by ($\tilde{4}$): There exists a non-null small set B such that $1_B(T^n 1_B/\Sigma_{i \leqq n} T^i 1_B)$ converges to zero in measure. There are analogous equivalent formulations of other conditions.

Finally, the invariant function, if it exists, is unique. The proof of uniqueness, given in [20], also applies in present conditions.

3. A counterexample

We now give an example of a transition measure which induces an L_1-operator T having a σ-finite (in fact even finite) invariant measure, but such that the boundedness condition (B) introduced in [20] fails; thus this condition is not necessary. Recall

$$\text{(B)} \qquad \forall h \in L_\infty, \qquad \liminf |T^{*n} h| < \infty \qquad \text{a.e.}$$

The example will be constructed on a discrete measure space; in this setting both implications (3) $\not\Rightarrow$ (6′) and (3′) $\not\Rightarrow$ (6) have been proved in [12].

3.1. EXAMPLE: Let $(X, \mathcal{A}, m)$ be the set of integers with the counting measure m. Let (α_n) be a sequence of strictly positive numbers such that

$$\sum_{n \geqq 0} \alpha_n (n+1)^2 2^n < \infty, \ \sum_{n \geqq 0} \alpha_n (n+1)^2 2^{2n} = \infty, \quad \text{and} \quad \sum_{n \geqq 0} \alpha_n = 1.$$

(Take e.g. $(n+1)^{-4} 2^{-n}$, and normalize.) Let (p_{ij}) be the stochastic matrix defined for each $n \geqq 0$ by

$$p_{0,2^n} = \alpha_n; \quad p_{2^n,2^n+1} = p_{2^n+1,2^n+2} = \cdots = p_{2^{n+1}-2,2^{n+1}-1} = p_{2^{n+1}-1,0} = 1.$$

In words the process at zero takes on values 2^n with probability α_n, then moves deterministically to $2^n + 1, 2^n + 2, \cdots, 2^{n+1} - 1, 0$.

We show that there is a probability measure π such that $\pi P = \pi$. π has to satisfy the following relations:

For every $n \geqq 0$, $\Sigma_i \pi_i p_{i,2^n} = \pi_0 p_{0,2^n} = \pi_0 \alpha_n = \pi_{2^n}$; for every n and every j, $1 \leqq j < 2^n$, $\pi_{2^n+j-1} = \Sigma_i \pi_i p_{i,2^n+j} = \pi_{2^n+j}$. The value π_0 is determined by the equation

$$\sum_{k \geqq 0} \pi_k = \pi_0 + \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \pi_{2^n+j} = \pi_0 \left(1 + \sum_{n=0}^{\infty} \alpha_n 2^n\right) = 1.$$

The vector (u_n) defined by:

$$u_0 = 1, \quad u_{2^n} = u_{2^n+1} = \cdots = u_{2^{n+1}-1} = (n+1)^{-2} 2^{-n}$$

satisfies the two relations:

$$\forall i, \qquad \sum_j p_{ij} u_i / u_j \leqq \sum_n \alpha_n (n+1)^2 2^n; \qquad \sum \frac{\pi_i}{u_i} = \infty.$$

Indeed $\Sigma_j p_{0j} u_0 / u_j = \Sigma_n \alpha_n u_0 / u_{2^n} = \Sigma_n \alpha_n (n+1)^2 2^n$; and if $i = 2^n + k$, $0 \leqq k < 2^n - 1$, then $\Sigma_j p_{ij} u_i / u_j = u_i / u_{i+1} = 1 \leqq \Sigma \alpha_n (n+1)^2 2^n$; if $i = 2^{n+1} - 1$, $\Sigma_j p_{ij} u_i / u_j = u_{2^{n+1}-1} / u_0 = 2^{-n} \leqq \Sigma \alpha_n (n+1)^2 2^n$. Furthermore, $\Sigma \pi_i / u_i = \pi_0 + \pi_0 \Sigma_n \alpha_n 2^{2n} (n+1)^2 = \infty$. Define a transition measure t by $t(i, \{j\}) = t_{ij} = p_{ij} u_i / u_j$.

The matrix (t_{ij}) induces an L_∞ operator V; indeed, for every i, $\Sigma_j t_{ij} \leqq \Sigma \alpha_n 2^n < \infty$. The condition (NS) is obviously satisfied by (t_{ij}), and since for every i, $\Sigma_j t_{ij} u_j = u_i \Sigma_j p_{ij} = u_i$, u is a fixed point of V. For every n, $t_{ij}^n = p_{ij}^n u_i / u_j$; hence t is irreducible, conservative and aperiodic because P is. The P-invariant probability π is such that $\pi_j = \lim_n p_{ij}^n$ for every i ([12]). Since $\Sigma \pi_j / u_j = \infty$, for every fixed i, $\lim_n \Sigma_j p_{ij}^n / u_j = \infty$, which implies that $V^n 1(i) = \Sigma_j t_{ij}^n$ converges to ∞. Thus (B) fails.

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